# Integer Factorization Problem in Cryptography

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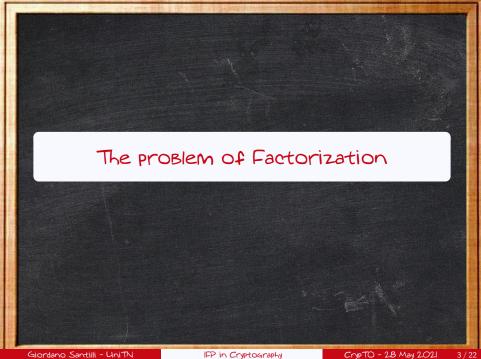
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IFP in Cryptography

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- The problem of Factorization
- 2 Public Key Encryption schemes based on IFP
- 3 Factorization Algorithms
- A pattern in successive remainders



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### Theorem (Fundamental Theorem of Arithmetic)

Every positive integer N greater than 1 can be represented in a unique way as a product of prime powers:

$$N = p_1^{e_1} \cdots p_k^{e_k},$$

where  $k \in \mathbb{N}^+$ ,  $p_1, \ldots, p_k$  prime numbers and  $e_1, \ldots, e_k \in \mathbb{N}$ .

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One-way problem:

$$\begin{array}{ccc} p_1^{e_1} \cdots p_k^{e_k} \xrightarrow{\text{easy}} N \\ & N \xrightarrow{\text{hard}} p_1^{e_1} \cdots p_k^{e_k} \end{array}$$

## Integer Factorization Problem (IFP)

#### Integer Factorization Problem (IFP)

Given a semiprime  $N \in \mathbb{Z}$ , find its prime factors p and q.

#### Remark

We call p the smaller factor and q the bigger one.

# Public Key Encryption schemes based on IFP



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- RSA (1976)
- Rabin Cryptosystem (1979)
- Goldwasser-Micali Cryptosystem (1982)
- Paillier Cryptosystem (1999)

### Generation of the key

- 1. Generate two random prime numbers p and q and compute N = pq;
- 2. Generate a random invertible  $e \in \mathbb{Z}_{\varphi(N)}$  and compute d such that  $ed \equiv 1 \mod \varphi(N)$ ;
- 3. (N,e) is the public key, while (p,q,d) is the private key.

#### Encryption

- 1. Consider a message  $m \in \mathbb{Z}_N$ ;
- 2. Compute and transmit  $c \equiv m^e \mod N$ .

#### Decryption

1. Compute 
$$c^d \equiv m^{ed} \equiv m \mod N$$
.



1) Given (N, e) and c is infeasible to recover m as  $\sqrt[e]{c \mod N}$ .



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**2** Given (N, e) is infeasible to recover d.

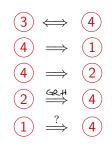


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    - ) Given N is infeasible to recover p and q.

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  - Given N is infeasible to recover  $\varphi(N)$ .
    - Given N is infeasible to recover p and q.



## Generation of the key

- 1. Generate two random prime numbers p and q such that  $p \equiv q \equiv 3 \mod 4$  and compute N = pq;
- 2. N is the public key, while  $\left(p,q\right)$  is the private key.

### Encryption

- 1. Consider a message  $m \in \mathbb{Z}_N$ ;
- 2. Compute and transmit  $c \equiv m^2 \mod N$ .

### Decryption

1. Solve the system

$$\begin{cases} m \equiv \pm \sqrt{c} \equiv \pm c^{\frac{p+1}{4}} \mod p \\ m \equiv \pm \sqrt{c} \equiv \pm c^{\frac{q+1}{4}} \mod q; \end{cases}$$

2. The original message m is one of the four solutions found.

# RaBin Cryptosystem

## Generation of the key

1. Generate two random prime numbers p and q such that  $p \equiv q \equiv 3 \mod 4$  and compute N = pq;

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2. The original message m is one of the four solutions found.

#### Security of Rabin cryptosystem

Recovering the plaintext m from the ciphertext c in the Rabin cryptosystem is as hard as finding a factorization for N.

## Goldwasser-Micali Cryptosystem

### Generation of the key

1. Generate two random prime numbers p and q and compute N = pq;

2. Generate 
$$x\in\mathbb{Z}_N$$
 such that  $\left(rac{x}{p}
ight)=\left(rac{x}{q}
ight)=-1;$ 

3.  $\left(N,x\right)$  is the public key, while  $\left(p,q\right)$  is the private key.

### Encryption

- 1. Consider a message  $\mathbf{m} = (m_1, \ldots, m_k) \in (\mathbb{Z}_2)^k$ ;
- 2. Generate random  $y_i \in \mathbb{Z}_N^*$  for  $1 \le i \le k$ ;
- 3. Compute  $c_i \equiv y_i^2 x^{m_i} \mod N$  and transmit  $\mathbf{c} = (c_1, \ldots, c_k) \in (\mathbb{Z}_N)^k$ .

#### Decryption

1. If  $c_i$  is a quadratic residue then  $m_i = 0$ , otherwise  $m_i = 1$ .

## Goldwasser-Micali Cryptosystem

#### Generation of the key

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- 2. Generate  $x \in \mathbb{Z}_N$  such that  $\left(\frac{x}{p}\right) = \left(\frac{x}{q}\right) = -1;$
- 3. (N, x) is the public key, while (p, q) is the private key.

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## Security of Goldwasser-Micali Cryptosystem

This algorithm is based on the quadratic residuosity problem (QRP): given (N, x) is computationally infeasible to decide whether x is a quadratic residue or not.

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$$\begin{array}{c} \mathsf{IFP} \Longrightarrow \mathsf{QRP} \\ & \overset{?}{\Longrightarrow} \mathsf{IFP} \end{array}$$

# Paillier Cryptosystem

#### Generation of the key

- 1. Generate two random prime numbers p and q and compute N=pq and  $\lambda=\operatorname{lcm}(p-1,q-1);$
- 2. Choose a random  $g \in \mathbb{Z}_{N^2}^*$  and compute

$$u \equiv \left(\frac{\left(g^{\lambda \mod N^2}\right) - 1}{N}\right)^{-1} \mod N;$$

3. (N,g) is the public key, while  $(p,q,\lambda,\mu)$  is the private key.

#### Encryption

- 1. Consider a message  $m \in \mathbb{Z}_N$ ;
- 2. Generate a random  $r \in \mathbb{Z}_N^*$  and compute  $c \equiv g^m \cdot r^N \mod N^2$ .

#### Decryption

1. Compute 
$$m \equiv \left(\frac{(c^{\lambda} \mod N^2) - 1}{N}\right) \cdot \mu \mod N.$$

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## Homomorphic Properties

Paillier encryption is homomorphic:

 $\mathsf{Decrypt}(\mathsf{Encrypt}(m_1) \cdot \mathsf{Encrypt}(m_2)) \equiv m_1 + m_2 \mod N.$ 

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#### Security of Paillier Cryptosystem

Paillier Cryptosystem is based on the composite residuosity problem (CRP): given (N, x), it is computationally infeasible to decide whether there exists  $y \in \mathbb{Z}_{N^2}$  such that  $x \equiv y^N \mod N^2$ .

### Homomorphic Properties

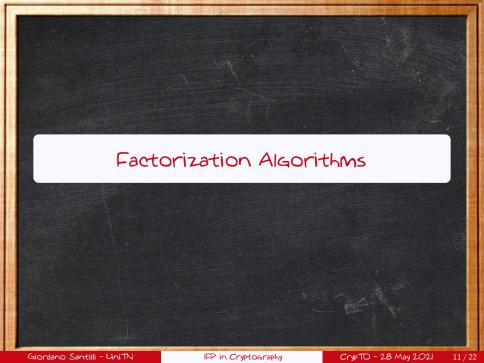
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```
IFP \implies CRPRSA \implies CRPCRP \stackrel{?}{\implies} IFP
```



Suppose we want to recover p and q from N.

#### Brute Force Algorithm

- 1. For any prime  $s \in \mathbb{P}$  starting from 2 check if  $N \equiv 0 \mod s$ ;
- 2. Stop when p is found, then  $q = \frac{N}{p}$ .

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Since p < q then  $p \leq \lfloor \sqrt{N} \rfloor$ , meaning that we have to check, in the worst case,  $\pi(\sqrt{N}) \sim \frac{\sqrt{N}}{\log \sqrt{N}} \sim O\left(\sqrt{N}\right)$  values.

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#### Effectiveness

This method is called Trial Division. It works best when p is small.

#### First-Category Algorithms

- These methods return the smaller prime divisor p of N.
- They are effective if  $p \approx 7 40$  digits.

First Category Algorithms			
Factorization Method	Execution Time		
Trial Division	$O\left(N^{\frac{1}{2}}\right)$		
Pollard's $p-1$ Algorithm	$O\left(N^{\frac{1}{2}}\right)$		
Pollard's $\rho$	$O\left(N^{\frac{1}{4}}\right)$		
Shanks' Class Group Method	$O\left(N^{\frac{1}{4}}\right)$		
Lenstra's Elliptic Curves Method (ECM)	$O\left(e^{\sqrt{2\log N \log \log N}}\right)$		

Table: Recap of some famous first category factorization methods for  $N = p \cdot q$ .

## Fermat's approach

IFP can be solved finding  $x,y\in\mathbb{Z}_N$  such that

$$x^2 \equiv y^2 \bmod N,$$

meaning that

$$N = pq|(x^2 - y^2) = (x - y)(x + y) \Longrightarrow p|(x - y)(x + y) \text{ and } q|(x + y)(x - y).$$

But since p and q are primes:

$$\begin{cases} p|(x-y) \lor p|(x+y) \\ q|(x-y) \lor q|(x+y) \end{cases}$$

# Fermat's method

The possible cases are the following:

$p \mid (x - y)$	$p \mid (x+y)$	$q \mid (x - y)$	$q \mid (x+y)$	gcd(x-y,N)	$\gcd(x+y,N)$	Factorization
1	✓	<ul> <li>Image: A second s</li></ul>	1	N	N	×
1	1	1	×	N	p	<ul> <li>Image: A second s</li></ul>
1	$\checkmark$	×	1	p	N	<ul> <li>Image: A set of the set of the</li></ul>
1	×	1	<ul> <li>Image: A second s</li></ul>	N	q	1
1	×	✓	×	N	1	×
1	×	X	$\checkmark$	p	q	1
×	1	$\checkmark$	×	q	p	1
×	1	X	1	1	N	×
×	✓	✓	1	q	N	✓

Table: Output for  $x^2 \equiv y^2 \mod N$ .

It is possible to recover a successful factorization in 6 cases over  $9 \approx 66\%.$ 

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✓	$\checkmark$	×	1	p	N	<ul> <li>Image: A set of the set of the</li></ul>
1	×	✓	1	N	q	1
1	X	1	X	N	1	×
<ul> <li>✓</li> </ul>	×	X	<ul> <li>Image: A set of the set of the</li></ul>	<i>p</i>	q	<ul> <li>✓</li> </ul>
×	$\checkmark$	1	X	q	p	1
×	1	X	1	1	Ν	×
×	✓	✓	<ul> <li>Image: A second s</li></ul>	q	N	<ul> <li>Image: A second s</li></ul>

Table: Output for  $x^2 \equiv y^2 \mod N$ .

It is possible to recover a successful factorization in 6 cases over  $9 \approx 66\%$ . Adding the condition  $x \not\equiv \pm y \mod N$  it is always possible to recover a non-trivial factor of N.

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### Second-Category Algorithms

- Do not take into account the distance between p and q and the complexity only depends on the size of N.
- Are effective if N has more than  $\approx 100$  digits and no small factors.
- They are based on Fermat's idea.

Second Category Algorithms			
Factorization Method	Execution Time		
Lehman's method	$O\left(N^{\frac{1}{3}}\right)$		
Shanks' Square Forms Factorization (SQUFOF)	$O\left(N^{\frac{1}{4}}\right)$		
Dixon's Factorization Method	$O\left(e^{2\sqrt{2\log N \log \log N}}\right)$		
Continued Fractions Method (CFRAC)	$O\left(e^{\sqrt{2\log N \log \log N}}\right)$		
Multiple Polynomial Quadratic Sieve (MPQS)	$O\left(e^{\sqrt{\log N \log \log N}}\right)$		
General Number Field Sieve (GNFS)	$O\left(e^{\sqrt[3]{\frac{64}{9}\log N(\log\log N)^2}}\right)$		

Table: Recap of some second category factorization methods for  $N = p \cdot q$ .

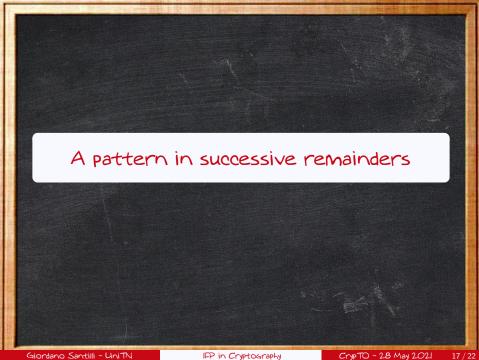
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# RSA Factoring Challenge (1991)

RSA-Number	Binary Digits	Date of Factorization	Method used
RSA-100	330	1 April 1991	MPQS
RSA-110	364	14 April 1992	MPQS
RSA-120	397	9 July 1993	MPQS
RSA-129	426	26 April 1994	MPQS
RSA-130	430	10 April 1996	GNFS
RSA-140	463	2 February 1999	GNFS
RSA-150	496	16 April 2004	GNFS
RSA-155	512	22 August 1999	GNFS
RSA-160	530	1 April 2003	GNFS
RSA-170	563	29 December 2009	GNFS
RSA-576	576	3 December 2003	GNFS
RSA-180	596	8 May 2010	GNFS
RSA-190	629	8 November 2010	GNFS
RSA-640	640	2 November 2005	GNFS
RSA-200	663	9 May 2005	GNFS
RSA-210	696	26 September 2013	GNFS
RSA-704	704	2 July 2012	GNFS
RSA-220	729	13 May 2016	GNFS
RSA-230	762	15 August 2018	GNFS
RSA-232	768	17 February 2020	GNFS
RSA-768	768	12 December 2009	GNFS
RSA-240	795	2 December 2019	GNFS
RSA-250	829	28 February 2020	GNFS

Table: Known factorizations of RSA moduli.

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Let 
$$m$$
 be  $\left\lfloor \sqrt{\frac{N}{2}} \right\rfloor \le m \le \left\lfloor \sqrt{N} \right\rfloor$  and let  
$$\begin{cases} N \equiv a_0 \mod m \\ N \equiv a_1 \mod (m+1) \\ N \equiv a_2 \mod (m+2), \end{cases}$$

where  $a_0$ ,  $a_1$ ,  $a_2$  are  $a_0 \le a_1 \le a_2$  or  $a_0 \ge a_1 \ge a_2$ . We define  $k := a_1 - a_0$  and

$$w := \begin{cases} a_2 - 2a_1 + a_0 & \text{if } a_2 - 2a_1 + a_0 \ge 0, \\ a_2 - 2a_1 + a_0 + m + 2 & \text{if } a_2 - 2a_1 + a_0 < 0. \end{cases}$$

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## Proposition

Let N be such that  $N \ge 50$  and let  $m \in \mathbb{N}^+$  with  $\left\lfloor \sqrt{\frac{N}{2}} \right\rfloor \le m \le \left\lfloor \sqrt{N} \right\rfloor$ , then  $w = \begin{cases} 2, \\ 4, \\ 6. \end{cases}$ 

#### Corollary

If there exists a value for 
$$m$$
 such that  $\left\lfloor \sqrt{\frac{N}{2}} \right\rfloor + 1 \le m \le \left\lfloor \sqrt{N} \right\rfloor - 1$ , then  $w = 4$ .

# Successive moduli

## Example

N = 925363 and $m = 680$ :	
$N \equiv a_0 = 563$	$\mathrm{mod}m$
$N \equiv a_1 = 565$	mod(m+1)
$N \equiv a_2 = 571$	mod(m+2)
$N \equiv 581$	mod(m+3)
$N \equiv 595$	mod(m+4)
$N \equiv 613$	mod(m+5)
$N \equiv 635$	mod(m+6)
$N \equiv 661$	mod(m+7)
$N \equiv 3$	mod(m+8)

# Successive moduli

## Example

N = 925363 and m = 680:

$$\begin{split} N &\equiv a_0 = 563 & \text{mod } m \\ N &\equiv a_1 = 565 = a_0 + k = 563 + 2 & \text{mod} (m+1) \\ N &\equiv a_2 = 571 = a_1 + k + w = 565 + 2 + 4 & \text{mod} (m+2) \\ N &\equiv 581 = 571 + 2 + 2 \cdot 4 & \text{mod} (m+3) \\ N &\equiv 595 = 581 + 2 + 3 \cdot 4 & \text{mod} (m+4) \\ N &\equiv 613 = 595 + 2 + 4 \cdot 4 & \text{mod} (m+4) \\ N &\equiv 635 = 613 + 2 + 5 \cdot 4 & \text{mod} (m+5) \\ N &\equiv 661 = 635 + 2 + 6 \cdot 4 & \text{mod} (m+7) \\ N &\equiv 3 = 661 + 2 + 7 \cdot 4 = 691 & \text{mod} (m+8) \end{split}$$

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## Proposition

Let 
$$N \ge 50$$
 and such that  $\left\lfloor \sqrt{\frac{N}{2}} \right\rfloor \le m \le \left\lfloor \sqrt{N} \right\rfloor$ , then for every  $i \in \mathbb{N}$ ,  

$$N \equiv \left( a_0 + ik + w \cdot \frac{i(i-1)}{2} \right) \mod (m+i).$$

#### Corollary

If 
$$\left\lfloor \sqrt{\frac{N}{2}} \right\rfloor + 1 \le m \le \left\lfloor \sqrt{N} \right\rfloor - 1$$
, then for every  $i \in \mathbb{N}$ ,  
 $N \equiv \left( a_0 + ik + 2i^2 - 2i \right) \mod (m+i).$ 

Consider the polynomial  $f \in \mathbb{Q}[x]$  of degree 2, such that

$$\begin{cases} f(0) = a_0, \\ f(1) = a_1, \\ f(2) = a_2. \end{cases}$$

## Proposition

Let 
$$\left\lfloor \sqrt{\frac{N}{2}} \right\rfloor + 1 \le m \le \left\lfloor \sqrt{N} \right\rfloor - 1$$
. Then, the interpolating polynomial  $f \in \mathbb{Q}(x)$  is such that, for every  $i \in \mathbb{Z}$ ,

$$N \equiv f(i) \bmod (m+i).$$

## Successive moduli in factorization

In order to find a factor of N, we would like to solve the following equation for some  $x \in \mathbb{Z}$ :

$$a_0 + ik + 2i^2 - 2i = x(m+i).$$

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#### Proposition

Let N be a semiprime and m such that  $\left\lfloor \sqrt{\frac{N}{2}} \right\rfloor + 1 \le m \le \left\lfloor \sqrt{N} \right\rfloor - 1$ . Then producing the factorization of N is equivalent to finding an integer  $i \in \mathbb{N}^+$  for which

$$N \equiv \left(a_0 + ik + 2i^2 - 2i\right) \equiv 0 \mod (m+i).$$

If we consider the interpolating polynomial f, then if m is close to one of the factor of N, then the roots of f are exactly the  $i \in \mathbb{Z}$  such that

 $f(i) \equiv 0 \bmod (m+i).$ 

However to achieve this result, we need to choose the first remainder  $a_0$  in the monotonic descending sequence that leads to 0.

#### Example

 ${\cal N}=925363$  and m=943, then

 $\begin{cases} N \equiv 280 \mod 943, \\ N \equiv 243 \mod 944, \\ N \equiv 208 \mod 945. \end{cases}$ 

The interpolating polynomial is

$$f(i) = i^2 - 38i + 280,$$

which has two roots:  $i_1 = 10$  and  $i_2 = 28$ . Therefore the two factors of N are:

$$m + i_1 = 953$$
  $m + i_2 = 971.$ 

# THANK YOU FOR THE ATTENTION!

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