Knot-based Key Exchange Protocol

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joint work with Arno Wildi

CrypTO Seminars, Politecnico di Torino

March 22nd, 2024

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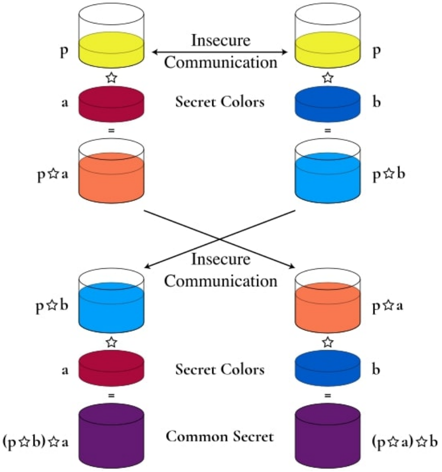

[Picture from Borradaile, G. "Defend Dissent." Corvallis: Oregon State University, 2021.]

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The secret common key is $g^{ba} = g^{ab}$.

- Diffie-Hellman Problem (DHP): Let G be a finite cyclic group and let g be a generator. Given g^a and g^b, find g^{ab}.

Group actions

Given G an abelian group with identity element e and a set X, a group action of G on X is a map

$$\begin{aligned} \star \colon G \times X &\longrightarrow X \\ (g, x) &\mapsto g \star x \end{aligned}$$

s.t. $e \star x = x$ and $g \star (h \star x) = (gh) \star x$ for all $g, h \in G$ and $x \in X$.

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Example: Let X be a cyclic finite group of order p and $G = \mathbb{Z}_p^\times$. Then

$$\begin{aligned} \mathbb{Z}_p^\times \times X &\longrightarrow X \\ (n, x) &\mapsto x^n \end{aligned}$$

is an **action** of $\mathbb{Z}_p^\times$ over X.

Generalised DHKE

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- Diffie-Hellman Group Action Problem (DHGAP): Let G, X and $\star$ as above. Given $x, y, z \in X$ such that $y = g \star x$ and $z = h \star x$ for some $g, h \in G$, find $(gh) \star x$.

Semigroups and semigroup actions

A semigroup is a set S together with a *binary operation* $\cdot : S \times S \to S$ that satisfies the associative property.

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Given S an abelian semigroup and a set X, an S-action on X (or a semigroup action of S on X) is a map

$$\begin{aligned} \star : S \times X &\longrightarrow X \\ (s, x) &\mapsto s \star x \end{aligned}$$

s.t. $s \star (r \star x) = (s \cdot r) \star x$ for all $s, r \in S$ and $x \in X$

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• Diffie-Hellman Semigroup Action Problem (DHSAP): Let S, X and $\star$ as above. Given $x, y, z \in X$ such that $y = s \star x$ and $z = r \star x$ for some $s, r \in S$, find $(gh) \star x$.

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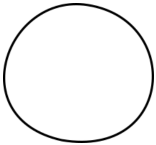
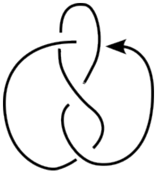


Unknot $\mathcal{U}$

Trefoil knot



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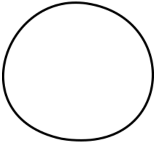

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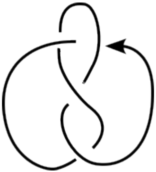


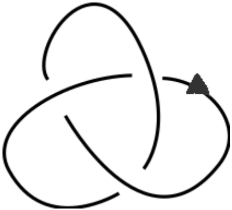
Oriented Figure-Eight knot

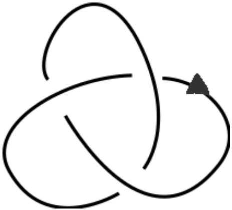
N.B.: We will consider just oriented knots.

Connected sum

Given two oriented knots K and K', we can define the *connected sum* $K \# K'$: cut the two knots and glue the corresponding ends (given by the orientation).

Example:





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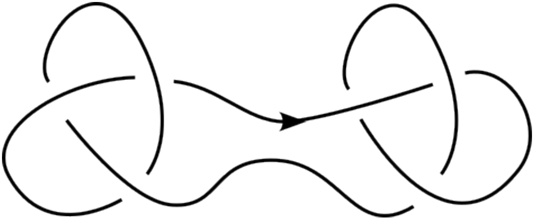

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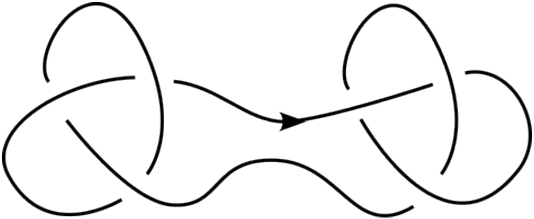

N.B.: With this operation, the set of oriented knots forms an abelian semigroup: $(\mathbf{oKnots}, \#, \mathcal{U})$.

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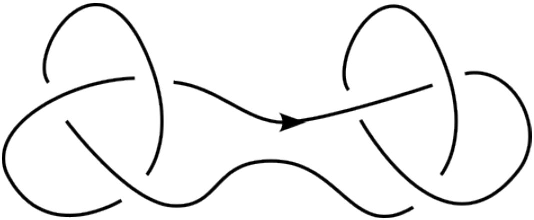
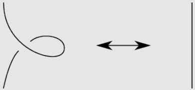


- Decomposition Problem: Given a knot K, find its prime decomposition $K = K_1 \# \cdots \# K_n$.

Definitions

Theorem (Reidemeister):

Two knots are the same if and only if they are related by a finite sequence of the Reidemeister moves:



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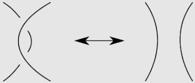
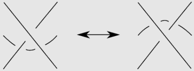


R2



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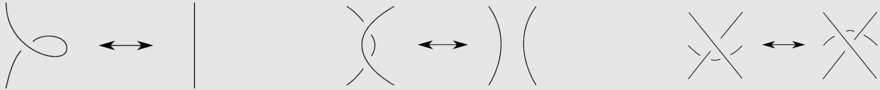

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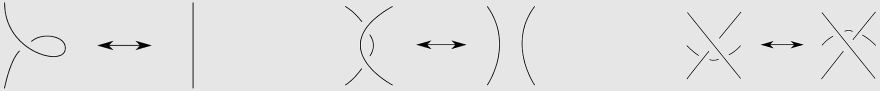

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↑ This is a hard mathematical problem. ↑

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To classify knots, one studies knot invariants, which are functions that do not change under Reidemeister moves.

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Fact: A finite type invariant of type d can be computed in

$$\mathcal{O}(c^d),$$

where c is the number of crossings of the knot.

Fixed a $d \in \mathbb{N}$, we can choose between <u>several distinct</u> finite type invariants of type d.

d	0	1	2	3	4	5	6
# d-Finite type invariants	1	1	2	3	6	10	19

d	7	8	9	10	11	12
# d-Finite type invariants	33	60	104	184	316	548

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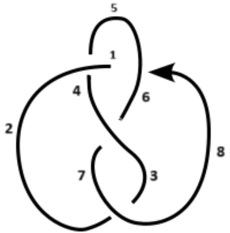

[X[4,1,5,2], X[2,8,3,7], X[6,4,7,3], X[8,5,1,6]]

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We need to "complicate" $A\#K$ and $B\#K$, in order to make them *unrecognisable*.

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Final idea

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Knot-based Key Exchange (final version)

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The secret common key is $V(A\#B\#K) = V(B\#A\#K)$.

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Remarks:

[1] https://github.com/denizkutlay/Randomeisterrandomeister, D. Kutluay

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- Underlying mathematical problem: Given $V(K)$, $V(A\#K)$ and $V(B\#K)$, find $V(A\#B\#K)$.

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- Recall that $(\mathbf{oKnots}, \#, \mathcal{U})$ is an *abelian semigroup*. Moreover, $\mathcal{U}$ is the only invertible element.
- To apply random Reidemeister moves, we use the program *Randomeister*[1].

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   In general, you will obtain more than one correspondence, so you have to choose *another* invariant and restart.

Choice of parameters

Goal: choose n to reach a 128-bit security level $\rightsquigarrow$ $> 2^{128}$ operations

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It is enough to consider $K_1 \# K_2 \# K_3 \# K_4 \# K_5$ with K_i **prime** knots with 19 crossings, since

$$\#\{\text{prime knots with 19 crossings}\} \approx 3 \cdot 10^8$$

$$\Rightarrow \quad n = 95$$

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- Given a string of quaterns of integers, when it represents an encoded knot?
- No attempt has yet been made to implement our protocol.

Thanks for your attention!

(Submitted to Cryptology ePrint Archive)

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